The Falling Ladder Paradox Revisited

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The falling ladder problem is a well-known, related-rates problem that is famous for its paradoxical solution. In this problem, the top of a ladder of length L slides down a vertical wall as the base of the ladder moves away from the wall at a constant speed, v_0 (see Figure 1). Students are asked to find the speed of the top of the ladder at certain heights as it falls. The familiar paradox is that the related-rates approach to solving the problem eventually gives physically unrealistic solutions—the speed approaches infinity as the top approaches the ground.

The resolution of the paradox comes by recognizing that the top of a real ladder will inevitably come away from the wall at some point during its descent. The related-rates approach will accurately model the motion while the ladder is in contact with the wall. After the ladder disengages, however, it must be treated as a physical pendulum whose pivot moves along the ground.

A number of authors have shared their perspectives on the falling ladder problem [2, 3, 5, 7]. It is not our goal to improve upon their treatments of the paradox, but rather to take the solution a few steps further. In addition, we offer some experimental support for the solution and introduce several other related-rates "paradoxes" that are found in modern calculus textbooks.

1 Falling ladder models

We begin by summarizing the falling ladder models presented by Scholten and Simoson [7]. In contrast to their analysis, we take θ to be the angle the top of the ladder makes with the vertical ($0 \le \theta \le \pi/2$).

With the origin at the point of intersection of the wall and the ground, let $b_0 \ge 0$ be the initial x-coordinate of the base of the ladder. While the ladder is in contact with the wall, we have the following in-contact (IC) model:

$$\sin\theta = \frac{b_0 + v_0 t}{L}, \qquad \dot{\theta} = \frac{v_0}{L\cos\theta}, \qquad \ddot{\theta} = \frac{v_0^2 \sin\theta}{L^2\cos^3\theta},$$

where the dot denotes differentiation with respect to time t. Notice that $\dot{\theta} \to \infty$ as $\theta \to \pi/2^-$, as the paradox contends. When the ladder is away from the wall, falling as a physical pendulum, torque considerations and Newton's 2nd Law give the following not-in-contact (NIC) model:

$$\ddot{\theta} = \frac{3g\sin\theta}{2L},$$

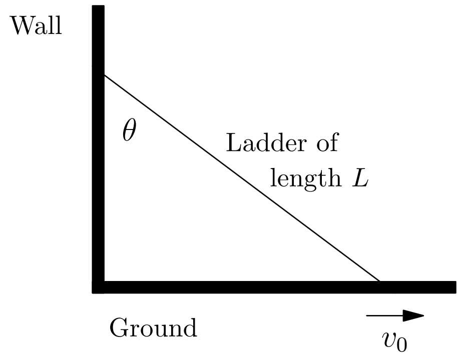

Figure 1: The falling ladder

where air resistance is being ignored and g is the acceleration due to the force of gravity. (The ladder's moment of inertia is taken to be $I = mL^2/3$. For details of the derivation of the NIC model, see the article by Kapranidis and Koo [5] or by Scholten and Simoson [7].)

As long as the angular acceleration in the NIC model is greater than the angular acceleration in the IC model, the ladder will remain in contact with the wall. At the moment the angular accelerations are equal, the ladder will disengage from the wall. Therefore the critical angle, θ_c, is the angle at which

$$\ddot{\theta}_{\mathrm{IC}} = \frac{v_0^2 \sin\theta}{L^2 \cos^3\theta} = \frac{3g\sin\theta}{2L} = \ddot{\theta}_{\mathrm{NIC}}.$$

Solving for θ gives

$$\theta_c = \cos^{-1}\left(\sqrt[3]{\frac{2v_0^2}{3gL}}\right),$$

which in turn gives the critical height

$$y_c = L\cos\theta_c = \sqrt[3]{\frac{2L^2 v_0^2}{3g}}. \tag{1}$$

Before the top of the ladder reaches the critical height y_c, the IC model applies. Once it has fallen to the height y_c, the NIC model takes over. If the initial height is less than y_c, or if $2v_0^2/(3gL) \geq 1$, the ladder will come away from the wall immediately as the motion starts.

At this point, previous articles on the falling ladder problem go on to discuss the numerical solution of the NIC model [7] or they proceed in other directions [2, 3, 5]. It seems to have gone unnoticed that a formula for $\dot{y}$, which is after all what the problem typically asks for, can be derived from the NIC model.

Letting $t_c = (L\sin(\theta_c) - b_0)/v_0$ be the time at which the critical angle is achieved, the NIC model initial-value problem is:

$$\ddot{\theta} = \frac{3g\sin\theta}{2L}, \qquad \theta(t_c) = \theta_c, \quad \dot{\theta}(t_c) = \frac{v_0}{L\cos\theta_c}.$$

The 2nd-order differential equation can be reduced to a 1st-order separable equation by means of the substitution $u = \dot{\theta}, u\,du/d\theta = \ddot{\theta}$. This gives

$$u\,\frac{du}{d\theta} = \frac{3g\sin\theta}{2L}, \quad u(\theta_c) = \frac{v_0}{L\cos\theta_c}.$$

After separating variables, integrating, and applying the initial condition, we find that

$$u(\theta) = \dot{\theta}(\theta) = \sqrt{\frac{v_0^2}{L^2}\sec^2\theta_c + \frac{3g}{L}(\cos\theta_c - \cos\theta)},$$

which reduces to

$$\dot{\theta} = \sqrt{A - B\cos\theta}, \tag{2}$$

where

$$A = \frac{3}{4}\left(\frac{12v_0 g}{L^2}\right)^{2/3} \qquad \text{and} \qquad B = \frac{3g}{L}.$$

Since the y-coordinate of the top of the ladder is given by $y = L\cos\theta$, it follows that $\dot{y} = -L\sin(\theta)\,\dot{\theta}$ or

$$\dot{y} = -L\sin\theta\,\sqrt{A - B\cos\theta}. \tag{3}$$

These formulas describe the ladder's motion when it is away from the wall, i.e., when $\theta_c \le \theta \le \pi/2$. As the top of the ladder hits the ground, its speed is given by

$$|\dot{y}(\pi/2)| = L\sqrt{A} = \sqrt{3}\,\sqrt[3]{3v_0 gL/2}. \tag{4}$$

If we are willing to integrate numerically, we can also determine when and where the top of the ladder will hit the ground. Indeed, equation (2) can be used to determine the time t required to reach the angle θ:

$$\int_{\theta_c}^{\theta} \frac{d\phi}{\sqrt{A - B\cos\phi}} = t - t_c.$$

This gives

$$t_{\text{end}} = t_c + \int_{\theta_c}^{\pi/2} \frac{d\phi}{\sqrt{A - B\cos\phi}},$$

and it follows that the top of the ladder will land at the point where its x-coordinate is

$$x_{\text{end}} = b_0 + v_0 t_{\text{end}} - L.$$

As one might expect, this expression for x_{end} can be written in a way that does not depend on the initial position of the ladder:

$$x_{\text{end}} = L\sin(\theta_c) - L + v_0 \int_{\theta_c}^{\pi/2} \frac{d\phi}{\sqrt{A - B\cos\phi}}. \tag{5}$$

Trial	L (m)	v_0 (m/s)	Measured y_c (m)	Theoretical y_c (m)	% Error
1	1.546	0.84	0.505	0.487	3.7%
2	1.546	0.80	0.490	0.472	3.8%
3	1.394	0.86	0.493	0.462	6.7%
4	1.394	0.80	0.459	0.440	4.3%
5	0.932	0.79	0.339	0.334	1.5%

Table 1: Typical results of the falling ladder experiment

2 Example and experiment

In their article, Scholten and Simoson [7] analyzed a falling ladder problem taken from a popular 1994 calculus textbook. We take our example from a 2010 edition of a different calculus text [8]. In this example, $L = 13\,\text{ft}$, $v_0 = 8\,\text{ft/s}$, and students are asked to find the speed of the top of the ladder when its height is 5 ft. Using equation (1), we find that the critical height, y_c, of the ladder is approximately 6.085 ft. Therefore, at the 5-foot mark, the ladder will be falling as a pendulum, and the expected, related-rates approach to the problem does not apply. Assuming that the initial height was greater than y_c, equations (3), (4), and (5) give

$$\dot{y}\big|_{y=5} \approx -18.375\,\text{ft/s}, \quad \dot{y}(\pi/2) \approx -29.602\,\text{ft/s}, \quad x_{\text{end}} \approx 0.670\,\text{ft}.$$

In contrast to 18.375 ft/s, the textbook gives 19.2 ft/s as the speed when $y = 5\,\text{ft}$.

As part of a project for a two-year college STEM competition, the authors of this article carried out a falling ladder experiment. Boards of various lengths were used as ladders, and a piece of chalk was affixed to the top of each board. With a board leaning against a wall, the bottom was pulled away at a constant rate as determined by a sonic ranging device. The chalk left a mark on the wall indicating where the top disengaged. For the purposes of the experiment, g was measured to be $(9.72 \pm 0.03)\,\text{m/s}^2$. Some typical results are shown in Table 1.

Although the details are not included here, we found that our results are within experimental error. Nonetheless, it is curious that the theoretical predictions consistently underestimated the observed values of y_c. A reasonable explanation for the discrepancies comes from air resistance. Because solid, rather wide, boards were used as ladders, the negligibility of air resistance is questionable. In the NIC model, air resistance would work to decrease $\ddot{\theta}$. This, in turn, would cause the intersection point of the graphs $\ddot{\theta}_{\text{IC}}$ and $\ddot{\theta}_{\text{NIC}}$ to shift left, decreasing θ_c and increasing y_c. To minimize the effect of air resistance, future experimenters may wish to use heavy ladders with small areas.

3 Other related-rates paradoxes

While the falling ladder paradox has generated a great deal of interest, there are other related-rates paradoxes that have received little, or no, attention. We close by mentioning three such problems, which we hope might provide inspiration for future projects. Even though these problems are quite common in textbooks, we cite only one reference for each as an example.

- *Melting snowball problem* [1]: In this problem, a melting sphere of ice loses volume at a constant rate. Students must find the rate of change of the radius when the sphere reaches a particular size. Paradoxically, the radius shrinks infinitely fast as the radius approaches zero. (A small bug on the surface would be in for quite a ride!) Some authors cleverly get around this paradox by placing a spherical iron ball in the center of their snowball [4].

- *Leaking conical tank* [9]: A downward pointing conical tank filled with water loses volume at a constant rate. Students must find the rate of change of water level when the water is at a particular depth. As above, the rate approaches $-\infty$ as the water level drops to zero.

- *Boat/bobber problem* [6]: In this problem, a boat or a fisherman's bobber is pulled in at a constant rate toward a dock above the water level. Students are asked to find the speed of the boat/bobber as it approaches the dock. As in the falling ladder problem, the related-rates approach predicts speeds tending to infinity. However, as any fisherman can attest, the floating object will come out of the water and swing as a shrinking pendulum at some point as it approaches the dock. (Incidentally, we believe that the boat in the example problem is too close to the dock for related rates to apply.)

Abstract

The related-rates falling ladder problem is well-known calculus exercise with a paradoxical twist. We revisit the problem and derive a closed-form solution. We also describe experimental support for the solution and discuss some related paradoxes that may tempt future experimenters.

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