A remarkably elementary proof of the irrationality of e

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The standard proofs of the irrationality of e make use of the infinite series representation

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{1}$$

or the corresponding alternating series representation for $1/e$. (One such proof is given at the end of this article.) While these proofs are elementary, they obviously require some familiarity with infinite series. The following proof requires only integration-by-parts and some basic properties of the Riemann integral. The sum (1) follows as a consequence, thereby making this proof useful as an introduction to infinite series.

e is irrational.

Proof: Suppose $e = a/b$, where a and b are positive integers. Choose an integer $n \geq \max\{b, e\}$. Now consider the definite integral $\int_0^1 e^{-x}\, dx$. This integral is easily evaluated to give $1 - \frac{1}{e}$. On the other hand, repeated integration-by-parts (n times) gives

$$1 - \frac{1}{e} = \frac{1}{e}\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) + \int_0^1 \frac{x^n}{n!} e^{-x}\, dx.$$

Upon multiplying both sides by e and isolating the integral, we obtain

$$e - 1 - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e\int_0^1 \frac{x^n}{n!} e^{-x}\, dx. \tag{2}$$

Multiplying both sides of (2) by $n!$ gives

$$n!(e-1) - n!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = e\int_0^1 x^n e^{-x}\, dx.$$

Because of the choice of n and the assumption that e is rational, the left hand side must reduce to an integer. However the value of the expression on the right is between zero and one. Indeed

$$0 < e\int_0^1 x^n e^{-x}\, dx \leq e\int_0^1 x^n\, dx = \frac{e}{n+1} < 1.$$

This contradiction implies that e must be irrational. ⋄

Notice that the integral in (2) approaches zero as $n \to \infty$. Therefore we obtain (1) as a by-product of the proof. The series representation (1) was derived in a similar way by Chamberland in [1] and by Johnson in [2].

A proof using the series for $1/e$ ...

Use the fact that

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!},$$

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and let S_n denote the nth partial sum of the series:

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

Notice that S_n is a rational number, and it can be written in the form $M/n!$, where M is an integer. By the alternating series estimation theorem, it follows that

$$S_n - \frac{1}{(n+1)!} < e^{-1} < S_n \text{ for even } n$$

and

$$S_n < e^{-1} < S_n + \frac{1}{(n+1)!} \text{ for odd } n.$$

In either case, e^{-1} is strictly between two rational numbers of the forms $\frac{a}{(n+1)!}$ and $\frac{a+1}{(n+1)!}$, where a is an integer. It follows that e^{-1} cannot be written as a fraction with denominator $(n+1)!$ for any $n \geq 0$. Since any rational number can be written as a fraction with denominator $(n+1)!$, we conclude that e^{-1} cannot be a rational number. Since $1/e$ is irrational, it follows that e is irrational. (This proof is similar to Sondow's geometric proof [3].)

References

[1] M. Chamberland, *The series for e via integration*, The College Mathematics Journal, 5 (1999), p. 397.

[2] W. Johnson, *Power series without Taylor's theorem*, The American Mathematical Monthly, 6 (1984), pp. 367–369.

[3] J. Sondow, *A geometric proof that e is irrational and a new measure of its irrationality*, The American Mathematical Monthly, 7 (2006), pp. 637–641.